Probabilistic extension of higher-order model-checking

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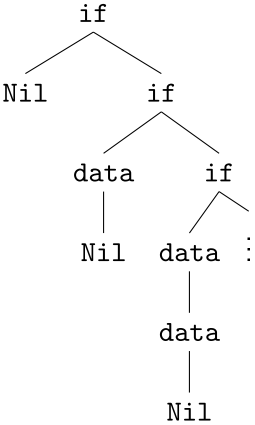
Roadmap

1. A quick introduction to higher-order model-checking (HOMC) and intersection types for HOMC
2. Automata for probabilistic properties, comparison with quantitative μ-calculus
3. Towards probabilistic HOMC: first steps and main challenges

Higher-order model-checking

Model-checking

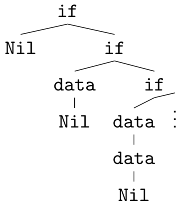
$\mathcal{T} \quad = $



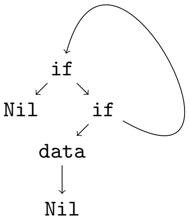
ϕ a logical property on trees, e.g. "all executions are finite".

Model-checking: does $\mathcal{T} \models \phi$?

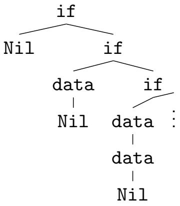
Finite representations of infinite trees



is not regular: it is not the unfolding of a finite graph as



Finite representations of infinite trees



but it is represented by a higher-order recursion scheme (HORS).

$$\mathcal{G} \;=\; \begin{cases} \mathtt{S} & = \quad \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = \quad \mathtt{if}\ x\,(\mathtt{L}\ (\mathtt{data}\ x\ )\ ) \end{cases}$$

Higher-order recursion schemes

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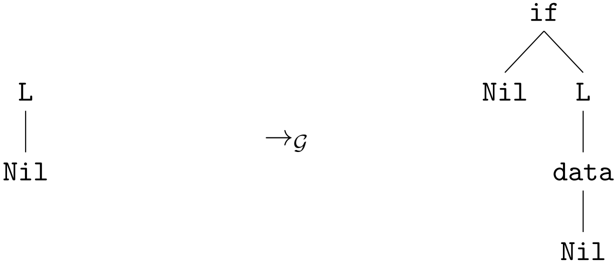
Rewriting starts from the start symbol S:

$$\mathtt{S} \quad \rightarrow_{\mathcal{G}} \quad \begin{array}{c} \mathtt{L} \\ | \\ \mathtt{Nil} \end{array}$$

Higher-order recursion schemes

$$\mathcal{G} \;=\; \left\{ \begin{array}{lcl} \mathtt{S} & = & \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = & \mathtt{if}\ x\ (\mathtt{L}\ (\mathtt{data}\ x\ )\ ) \end{array} \right.$$



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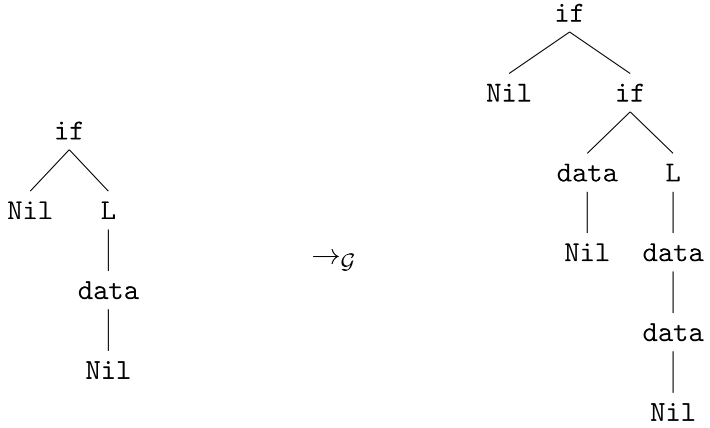

Higher-order recursion schemes

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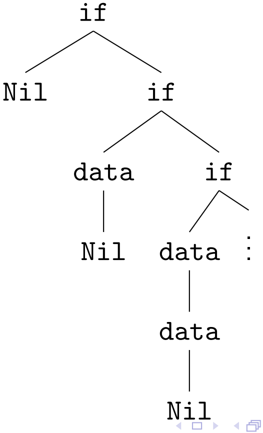
$\langle \mathcal{G} \rangle \quad =$



Higher-order recursion schemes

$$\mathcal{G} \;=\; \left\{ \begin{array}{lcl} \mathtt{S} & = & \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = & \mathtt{if}\ x\,(\mathtt{L}\ (\mathtt{data}\ x\,)\,) \end{array} \right.$$

HORS can alternatively be seen as simply-typed λ-terms with

simply-typed recursion operators $Y_\sigma \;:\; (\sigma \rightarrow \sigma) \rightarrow \sigma$.

Modal μ-calculus

Equivalent to MSO over trees.

$$\phi,\ \psi\ ::=\ X \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Box\,\phi \mid \Diamond_i\,\phi \mid \mu X.\,\phi \mid \nu X.\,\phi$$

$\Diamond_i\,\phi$: ϕ holds on a successor in direction i

$\Diamond\,\phi$: ϕ holds on a successor

$\Box\,\phi$: ϕ holds on all successors

Modal μ-calculus

Equivalent to MSO over trees.

$$\phi,\,\psi \;::=\; X \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Box\,\phi \mid \Diamond_i\,\phi \mid \mu X.\,\phi \mid \nu X.\,\phi$$

$\mu X.\,\phi$ is the least fixpoint of $\phi(X)$. It is computed by expanding finitely the formula:

$$\mu X.\,\phi(X) \quad\longrightarrow\quad \phi(\mu X.\,\phi(X)) \quad\longrightarrow\quad \phi(\phi(\mu X.\,\phi(X)))$$

$\nu X.\,\phi$ is the greatest fixpoint of $\phi(X)$. It is computed by expanding infinitely the formula:

$$\nu X.\,\phi(X) \quad\longrightarrow\quad \phi(\nu X.\,\phi(X)) \quad\longrightarrow\quad \phi(\phi(\nu X.\,\phi(X)))$$

Modal μ-calculus

Equivalent to MSO over trees.

$$\phi,\,\psi \ ::= \ X \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Box\,\phi \mid \ \Diamond_i\,\phi \mid \mu X.\,\phi \mid \nu X.\,\phi$$

Example formula:

$$\nu X.\,(\ \texttt{if} \wedge \Diamond_1\ (\ \mu Y.\,(\ \texttt{Nil} \vee \Box Y\ )\ ) \wedge \Diamond_2\ X\ )$$

Companion automata model: APT = ATA + parity condition.

Alternating tree automata (ATA)

ATA: non-deterministic tree automata whose transitions may duplicate or drop a subtree.

Typically: $\delta(q_0, \texttt{if}) \;=\; (2, q_0) \wedge (2, q_1)$.

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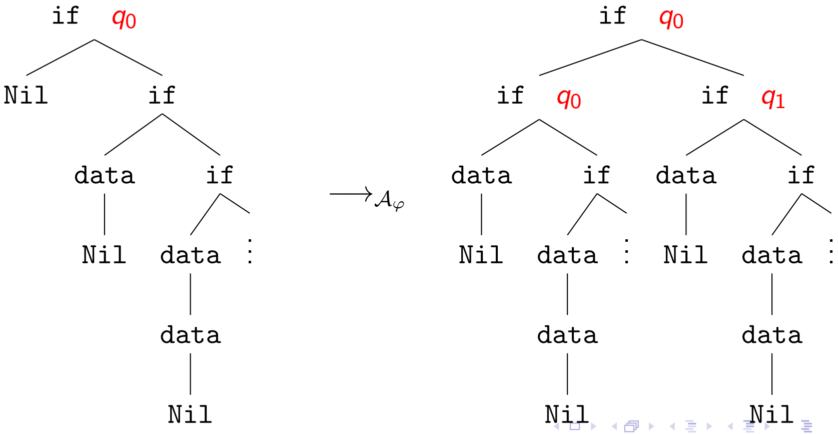
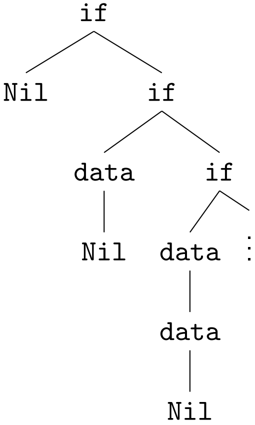


Alternating parity tree automata

Express reachability with ATA: does every branch ends by `Nil`?



Problem: ATA execute coinductively.

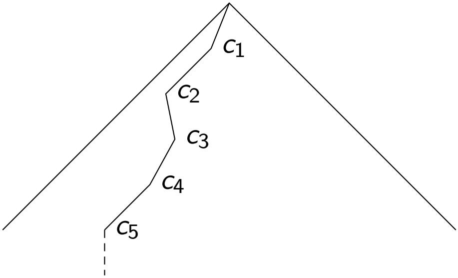
Solution: parity condition.

Alternating parity tree automata

Each state of an APT is attributed a color

$$\Omega(q) \in Col \subseteq \mathbb{N}$$

An infinite branch of a run-tree is winning iff the maximal color among the ones occuring infinitely often along it is even.



Alternating parity tree automata

Each state of an APT is attributed a color

$$\Omega(q) \in Col \subseteq \mathbb{N}$$

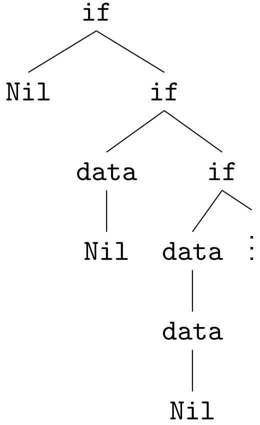
An infinite branch of a run-tree is winning iff the maximal color among the ones occuring infinitely often along it is even.

A run-tree is winning iff all its infinite branches are.

For a MSO formula φ:

$$\mathcal{A}_\varphi \text{ has a winning run-tree over } \langle \mathcal{G} \rangle \text{ iff } \langle \mathcal{G} \rangle \vDash \phi.$$

Alternating parity tree automata



$$Q = \{q\}$$

$$\Omega(q) = 1$$

$$\delta(\texttt{if}, q) = (1, q) \wedge (2, q)$$

$$\delta(\texttt{data}, q) = (1, q)$$

$$\delta(\texttt{Nil}, q) = \top$$

HOMC and intersection types

Alternating tree automata and intersection types

A key remark (Kobayashi 2009):

$$\delta(q_0, \mathtt{if}) \ = \ (2, q_0) \wedge (2, q_1)$$

can be seen as the intersection typing

$$\mathtt{if} \ : \ \emptyset \rightarrow (q_0 \wedge q_1) \rightarrow q_0$$

refining the simple typing

$$\mathtt{if} \ : \ o \rightarrow o \rightarrow o$$

Alternating tree automata and intersection types

A run-tree over $\mathtt{if} \quad T_1 \quad T_2$ is a derivation of $\emptyset \vdash \mathtt{if} \quad T_1 \quad T_2$:

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash \mathtt{if} : \emptyset \to (q_0 \wedge q_1) \to q_0}\,\delta \qquad \color{red}{\emptyset}}{\emptyset \vdash \mathtt{if}\ T_1 : (q_0 \wedge q_1) \to q_0}\,\mathrm{App} \qquad \dfrac{\vdots}{\color{red}{\emptyset \vdash T_2 : q_0}} \qquad \dfrac{\vdots}{\color{red}{\emptyset \vdash T_2 : q_1}}}{\emptyset \vdash \mathtt{if}\ T_1\ T_2 : q_0}\,\mathrm{App}$$

Intersection types naturally lift to higher-order – and thus to $\mathcal{G}$, which finitely represents $\langle \mathcal{G} \rangle$.

Theorem (Kobayashi)

$S : q_0 \vdash S : q_0$ *iff* *the ATA $\mathcal{A}_\varphi$ has a run-tree over $\langle \mathcal{G} \rangle$.*

A type-system for verification: without parity conditions

(G.-Melliès 2014, from Kobayashi 2009 and Kobayashi-Ong 2009)

Axiom $$\frac{}{x : \bigwedge_{\{i\}} \ \theta_i :: \kappa \ \vdash \ x : \theta_i :: \kappa}$$

δ $$\frac{\{ (i, q_{ij}) \mid 1 \leq i \leq n, 1 \leq j \leq k_i \} \quad \text{satisfies} \quad \delta_A(q, a)}{\emptyset \vdash a : \bigwedge_{j=1}^{k_1} \ q_{1j} \to \ldots \to \bigwedge_{j=1}^{k_n} \ q_{nj} \to q :: o \to \cdots \to o}$$

App $$\frac{\Delta \vdash t : (\ \theta_1 \ \wedge \cdots \wedge \ \theta_k) \to \theta :: \kappa \to \kappa' \qquad \Delta_i \vdash u : \theta_i :: \kappa}{\Delta + \Delta_1 + \ldots + \Delta_k \ \vdash \ t\, u : \theta :: \kappa'}$$

λ $$\frac{\Delta, x : \bigwedge_{i \in I} \ \theta_i :: \kappa \ \vdash \ t : \theta :: \kappa'}{\Delta \ \vdash \ \lambda x \, . \, t : \left(\bigwedge_{i \in I} \ \theta_i\right) \to \theta :: \kappa \to \kappa'}$$

fix $$\frac{\Gamma \vdash \mathcal{R}(F) : \theta :: \kappa}{\overline{F : \theta :: \kappa \vdash F : \theta :: \kappa}}$$

Colored intersection types

A type-system for verification

(G.-Melliès 2014, from Kobayashi-Ong 2009)

$$\text{App} \quad \frac{\Delta \vdash t : (\Box_{c_1}\ \theta_1\ \wedge \cdots \wedge \Box_{c_k}\ \theta_k) \rightarrow \theta\ ::\ \kappa \rightarrow \kappa' \qquad \Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\ +\ \Box_{c_1}\Delta_1\ +\ \ldots\ +\ \Box_{c_k}\Delta_k\ \ \vdash\ \ t\,u\ :\ \theta :: \ \kappa'}$$

$+$ coloring of typing tree nodes and parity condition on derivations

A type system for verification

Theorem (G.-Melliès 2014, from Kobayashi-Ong 2009)

$S : q_0 \vdash S : q_0$ admits a winning typing derivation

iff

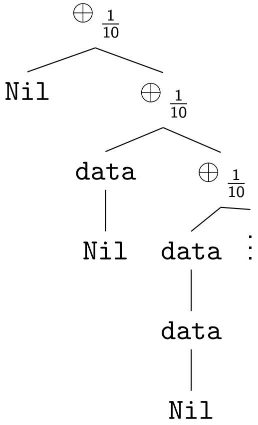
the alternating parity automaton $\mathcal{A}$ has a winning run-tree over $\langle \mathcal{G} \rangle$.

Static analysis: directly on the finite HORS $\mathcal{G}$.

Probabilistic automata

Probabilistic HOMC

```
IntList random_list() {
  IntList list = Nil;
  while(rand() > 0.1) {
    list := rand_int()::list;
  }
  return l;
}
```

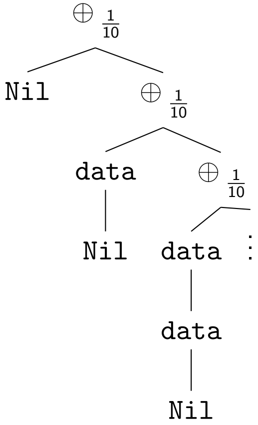


Probabilistic HOMC

Allows to represent probabilistic programs.

And to define higher-order regular MDP: those bisimilar to their encoding represented by a HORS.

(encoding of probabilities + payoffs in symbols)



Probabilistic automata

Idea: no longer verify ϕ but $Pr_{\geq p}\ \phi$.

- Step one: quantitative ATA.
- Step two: deal with colors and parity condition.

Probabilistic automata (PATA):

- ATA on non-probabilistic symbols
- $+$ probabilistic behavior on choice symbol $\oplus_p$

Run-tree: labels $(q,\ p_b,\ p_f)$.

The root of a run-tree of probability p is labeled $(q_0, 1, p)$, where p is the probability with which we want the tree to satisfy the formula.

Probabilistic alternating tree automata

Probabilistic behavior:

$$\oplus_p \quad (q,\, p_b,\, p_f)$$
$$b \qquad\qquad c$$

is labeled as

$$\oplus_p \quad (q,\, p_b,\, p_f)$$
$$b \quad (q,\, p \times p_b,\, p'_f) \qquad\qquad c \quad (q,\, (1-p) \times p_b,\, p_f - p'_f)$$

for some $p'_f \in [0, p_f]$ such that $p'_f \leq p \times p_b$ and $p_f - p'_f \leq (1-p) \times p_b$.

Example of PATA run

$\phi \quad = \quad$ "all the branches of the tree contain $\texttt{data}$"

is modeled by the PATA:

- $\delta_1(q_0, \texttt{data}) \;=\; (1, q_1),$
- $\delta_1(q_1, \texttt{data}) \;=\; (1, q_1),$
- $\delta_1(q_0, \texttt{Nil}) \;=\; \perp,$
- $\delta_1(q_1, \texttt{Nil}) \;=\; \top.$

Example of PATA run

- $\oplus_{\frac{1}{10}}$ $(q_0,\ 1,\ \frac{9}{10})$
  - Nil $(q_0,\ \frac{1}{10},\ 0)$
  - $\oplus_{\frac{1}{10}}$ $(q_0,\ \frac{9}{10},\ \frac{9}{10})$
    - data $(q_0,\ \frac{9}{100},\ \frac{9}{100})$
      - Nil $(q_1,\ \frac{9}{100},\ \frac{9}{100})$
    - $\oplus_{\frac{1}{10}}$ $(q_0,\ \frac{81}{100},\ \frac{81}{100})$
      - data $(q_0,\ \frac{81}{1000},\ \frac{81}{1000})$
        - data $(q_1,\ \frac{81}{1000},\ \frac{81}{1000})$
          - Nil $(q_1,\ \frac{81}{1000},\ \frac{81}{1000})$
      - $\vdots$

Intersection types for PATA

As for ATA, except for tree constructors:

$$\frac{\{\,(i, q_{ij}) \mid 1 \leq i \leq n, 1 \leq j \leq k_i\} \quad \text{satisfies} \quad \delta_A(q, a)}{\emptyset \vdash a \,:\, \bigwedge_{j=1}^{k_1} (q_{1j}, p_b, p_f) \to \ldots \to \bigwedge_{j=1}^{k_n} (q_{nj}, p_b, p_f) \to (q, p_b, p_f)}$$

$$\frac{p'_f \in]0, p_f[\quad \text{and} \quad p'_f \leq p \times p_b \quad \text{and} \quad p_f - p'_f \leq (1-p) \times p_b}{\emptyset \vdash \oplus_p \,:\, (q, p \times p_b, p'_f) \to (q, (1-p) \times p_b, p_f - p'_f) \to (q, p_b, p_f)}$$

$$\frac{q \in Q \quad \text{and} \quad p \times p_b \geq p_f}{\emptyset \vdash \oplus_p \,:\, (q, p \times p_b, p_f) \to \emptyset \to (q, p_b, p_f)}$$

$$\frac{q \in Q \quad \text{and} \quad (1-p) \times p_b \geq p_f}{\emptyset \vdash \oplus_p \,:\, \emptyset \to (q, (1-p) \times p_b,\ p_f) \to (q, p_b, p_f)}$$

Intersection types for PATA

Theorem

$$\emptyset \vdash S : (q_0, 1, p)$$

iff

the PATA $\mathcal{A}$ has a run-tree of probability p over the tree $\langle \mathcal{G} \rangle$ generated by $\mathcal{G}$.

Under connection Rel/non-idempotent types, we obtain a similar denotational theorem.

Note that $[\![o]\!] \;=\; Q \times [0,1] \times [0,1]$.

PATA and quantitative μ-calculus

Quantitative μ-calculus (McIver-Morgan): interpret ϕ not in $\mathbb{B}$ but in $[0, 1]$.

When all payoffs are 1, semantics = size of the set of branches satisfying ϕ:

$$\|\phi\|_{\mathcal{V}}^{\underline{\sigma},\bar{\sigma}} \cdot s \;=\; \int_{[\![\phi]\!]_{\mathcal{V}}^{\underline{\sigma},\bar{\sigma}} \cdot s} Val$$

Result holding for regular (= finite) Markov chains.

PATA and quantitative μ-calculus

Deal with infinite branches? PATA accept them all...

For trivial formulas (only ν, never μ/only color is 0 = safety properties) and all payoffs set to 1 (for commodity, can be patched):

$$\|\phi\|_{\mathcal{V}} \quad = \quad \sup\{p \in [0,1] \mid \text{there is a run-tree of probability } p\}$$

PATA acts similarly to the game interpretation, resolving non-determinism but playing all alternating choices in parallel.

The type system approach captures these safety properties.

How to capture the general parity condition?

Towards the parity condition

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Idea: a colored run-tree of probability

$$p - p_{bad}$$

is

- a run-tree of probability p,
- where p_{bad} is the measure of the set of rejecting ($=$ odd-colored) branches in the run-tree.

Problem: relate the size of rejecting branches set throughout infinite β-reduction?

Thank you for your attention!

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