A semantic interpretation of tree automata

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joint work with Paul-André Melliès

PPS & LIAFA

October 11th 2013

Introduction

- A problem in verification : model-checking of MSO over trees produced by higher-order recursion schemes
- It is decidable (Ong 2006)
- Many connections with semantics appear in this result
- Our aim: obtaining this decidability result by semantic means

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Higher-order recursion schemes

- Models of recursive programs used in verification since the 60's
- Informally : we have a ranked alphabet Σ, non-terminals, variables, an axiom and parametrized rewriting rules
- Example : $\Sigma = \{a : 2,\ b : 1,\ c : 0\}$.

Higher-order recursion schemes : an example

$$\begin{array}{c} S \rightarrow F\ c \\ F\ x \rightarrow a\ x\ (F\ (b\ x)) \end{array}$$

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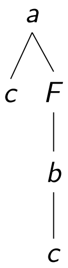

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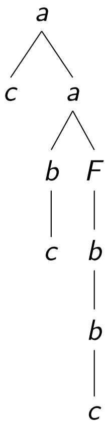

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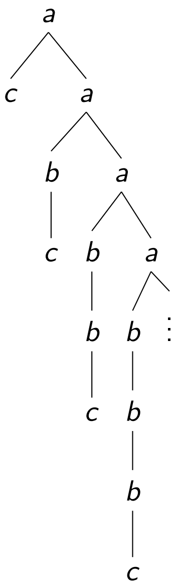

λ-calculus

- λ-calculus is a calculus of functions
- It is built from variables and constants by abstraction and application
- Grammar : $t \ ::= \ x \mid \lambda x.t \mid t_1 \, t_2$

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$$\lambda f.\, f\,(\,f\,(\,a\,)\,)$$

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Axiom $$\frac{}{\Gamma\,,\,x : A \;\vdash\; x : A}$$

Application $$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Abstraction $$\frac{\Gamma, x : A \;\vdash\; M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}$$

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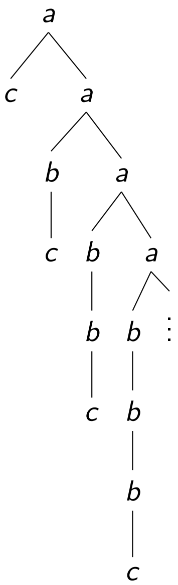

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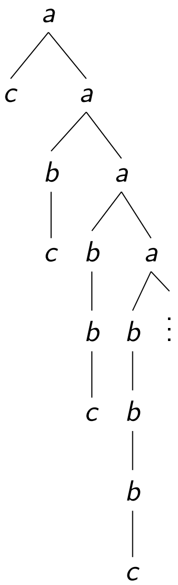

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- We use recursion schemes to build trees corresponding to all the possible behaviours of a progam (= a λ-term)
- We want to express MSO properties over them
- Over such trees :

$$\text{MSO} \Leftrightarrow \text{modal } \mu\text{-calculus} \Leftrightarrow \text{alternating parity tree automata}$$

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Alternating tree automata

- A kind of (top-down) tree automata...
- ...whose run-trees have a very special shape
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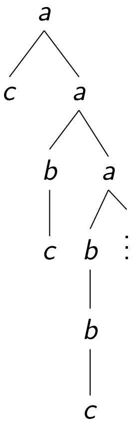
δ maps a state and a Σ symbol to a conjunction of states to label each son :

Example : $\delta(q_0, a) \;=\; (1, q_1) \wedge (2, q_0) \wedge (2, q_2)$

The automaton duplicates the right son, runs with state q_0 on a copy and q_2 on the other, and runs with q_1 on the left son.

Alternating tree automata: example of an execution



We start from state q_0.

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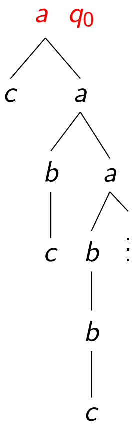

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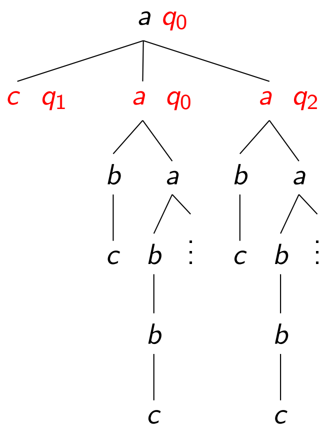

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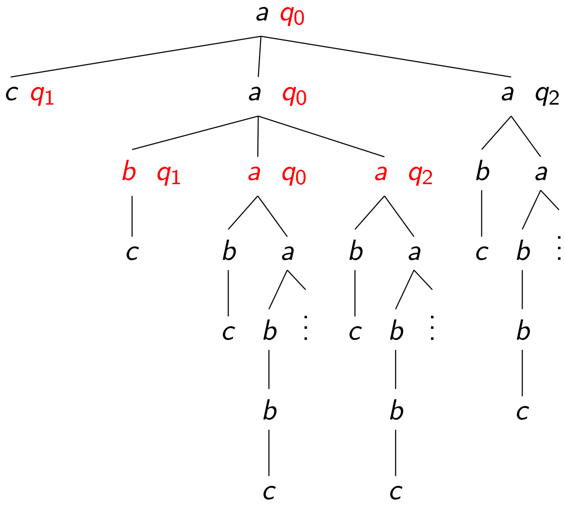

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Alternating tree automata and typing

Another point of view : δ gives a way to type the symbols of Σ.

We had $a : o \Rightarrow o \Rightarrow o$.

We will have $a : q_1 \Rightarrow (q_0 \wedge q_2) \Rightarrow q_0$: we refine types by interpreting the base type o with Q.

This extends to terms, and thus to HORS rules, for example :

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Model checking thus amounts to:

- Computing the biggest type for every non-terminal, which has to be consistent with rewriting
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Meaning : a (general) function has the power to duplicate or drop every of its arguments, before using all the copies linearly.

Point of view of resource management.

If we refine o with Q, an element of $!o$ is a *collection* of states.

An element of $!o \multimap o$	$=$	several states producing exactly one state.
	$=$	an intersection of states returning a state
	$=$	the alternating behaviour of a unary symbol

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Models, terms and types

- In a model, a type is interpreted as a mathematical object
- For example, in a qualitative model,

$$!o \multimap o \; = \mathcal{P}_{fin}(Q) \times Q$$

- A term is then interpreted as a subset of (the interpretation of) its type.
- Here : we can think of the interpretation of a term as the set of all its possible intersection types.
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We obtained two different linear typing systems for HORS.

They differ on their answer to the question: what is a *collection* of states ?

- A (coherent) set of elements of Q ?
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We have obtained the same result in the quantitative model Rel.

Moreover, with our type system reflecting Rel, we can describe a run over a term only by a derivation in (indexed) linear logic, which corresponds to the resource management of the automaton.

We can thus forget the term and only study an object which reflects how its behaviour affects states of an automaton.

Our next concerns

First of all, we need to deal properly with recursion and generation of infinite trees.

Then we will study the connection between:

- Rel, which reflects the (infinitary) execution of the automaton
- ScottL, the finitary model in which we can decide the existence of an alternating run

Ehrhard (2012) : "ScottL is Rel where you do not count the multiplicities"

How do this semantically explain the decidability result ?

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